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# The Gauged Vector Model in Four-Dimensions: Resolution of an Old Problem?

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**Abstract:** A calculation of the renormalization group improved effective potential for the gauged  $U(N)$  vector model, coupled to  $N_f$  fermions in the fundamental representation, computed to leading order in  $1/N$ , all orders in the scalar self-coupling  $\lambda$ , and lowest order in gauge coupling  $g^2$ , with  $N_f$  of order  $N$ , is presented. It is shown that the theory has two phases, one of which is asymptotically free, and the other not, where the asymptotically free phase occurs if  $0 < \lambda/g^2 < \frac{4}{3}(\frac{N_f}{N} - 1)$ , and  $\frac{N_f}{N} < \frac{11}{2}$ . In the asymptotically free phase, the effective potential behaves qualitatively like the tree-level potential. In the other phase, the theory exhibits all the difficulties of the ungauged ( $g^2 = 0$ ) vector model. Therefore the theory appears to be consistent (only) in the asymptotically free phase.

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## 1. Introduction

The search for non-perturbative methods in quantum field theory remains a central issue of the subject. Although great progress has been made recently using duality [1], there is still considerable interest in other approaches to strong-coupling questions, particularly as the new methods are limited to supersymmetric theories at present. One of the other techniques most frequently considered is the  $1/N$  expansion for a theory with internal symmetry, continued to  $O(N)$  or  $U(N)$  for example. Applications include 't Hooft's analysis of gauge theories [2], the demonstration of string behavior in two-dimensional QCD [3], and of  $O(N)$  invariant  $\lambda\phi^4$  theory [4, 5].

The  $1/N$  expansion for  $\lambda\phi^4$  theory (in 3+1 dimensions) with  $O(N)$  symmetry (the so-called vector model) has been extensively studied as a renormalized field theory [4, 5]. However, the renormalized vector model encounters a number of problems [5]. Among these are:

- (1) The effective potential of the theory is double-valued, where the lower branch of the potential exhibits unbroken internal symmetry at its minimum, *i.e.*,  $\langle\phi_a\rangle = 0$  [See Fig. 3 of Abbott, *et al.*, Ref. 5.] This phase is tachyon-free in all orders of the  $1/N$  expansion. The upper branch of the effective potential does allow a spontaneous broken symmetry, but at the expense of the appearance of tachyons, which signals a decay to the lower energy phase. In higher orders of the  $1/N$  expansion, the upper branch of the effective potential becomes everywhere complex.
- (2) The effective potential has no lowest energy bound as the external field  $\phi \rightarrow \infty$ . The tachyon-free phase (*i.e.*, with  $\langle\phi_a\rangle = 0$ ) tunnels non-perturbatively to this unstable vacuum.

The primary motivation of this paper is to provide a plausible resolution of the problems encountered by Abbott, *et al.* [5], although there may be other solutions as well. It is probably relevant that the ungauged  $\lambda\phi^4$  theory in four dimensions seems to be trivial [8], but this feature

is not the focus of this paper.

These difficulties make the renormalized vector model, evaluated in the  $1/N$  expansion, unsuitable for phenomenology. There was also interest in studying this model in the double-scaling limit [6], where one considers the correlated limit,  $N \rightarrow \infty$  and  $\lambda \rightarrow \lambda_c$ , where  $\lambda_c$  is a critical value of the coupling. Unfortunately, just at the critical point, the effective potential becomes everywhere complex [8], so that particular application of the vector model is also not possible.

One response to these problems is to consider a cutoff version of the vector model in the  $1/N$  expansion [9, 10]. In that case a viable phenomenology, with spontaneously broken symmetry and no tachyons, does exist to leading order in  $1/N$ . [Unfortunately, a double-scaling limit is still not possible even in the cutoff-version of the vector model [10, 11].] Since a cutoff mass  $\Lambda$  represents an energy scale above which the scalar fields have significant interactions with other degrees of freedom, the cutoff  $\lambda\phi^4$  theory cannot be regarded as a closed system, in that there are degrees of freedom which have been neglected, and in some sense incorporated into the cutoff  $\Lambda$ .

The question is then how should one couple the scalar fields to additional degrees of freedom, so that the system is consistent with just these degrees of freedom and no cutoff? In this paper, we argue that one way this can be accomplished in the  $1/N$  expansion is by gauging the scalar fields, and adding  $N_f$  massless fermions in the fundamental representation, if the scalar self-coupling satisfies  $0 < \lambda < \frac{4}{3}(\frac{N_f}{N} - 1)g^2$  and  $N_f/N < 11/2$  in the large  $N$  limit. If not, one returns to all the difficulties of the ungauged model. Notice that the model is **not** asymptotically free for  $N_f = 0$  in the large  $N$  limit.

The gauged vector model in the  $1/N$  expansion was previously considered by Kang [12]. However, his calculation is inadequate for our purposes, as the renormalization scheme chosen would not allow for a conventional Higgs mechanism. More importantly, the conclusions drawn

by Kang were not reliable [13], as they depended on features of the effective potential outside the domain of validity of the calculation, as evidenced by large logarithms. In our work we remedy both of these difficulties.

In Sec. 2 we formulate the gauged vector model coupled to fermions in the fundamental representation, and solve for the effective potential to leading orders in  $1/N$ , to all orders in  $\lambda$ , and to leading order in  $g^2$ . A renormalization group improved effective potential is constructed and analyzed in Sec. 3. It is argued that the theory has two phases. If  $\lambda/g^2$  is small enough, the model is asymptotically free and the theory is consistent, in that the difficulties found by Abbott, *et al.*, [5] are absent in this phase. In Sec. 4 there is a brief discussion of the issue of gauge invariance.

## 2. The Gauged Vector Model

Let us consider a renormalizable theory of gauged complex scalar fields in the fundamental representation of  $U(N)$ , with gauged-fixed Lagrangian density [12], and  $N_f$  massless fermions in the fundamental representation as well.

$$\begin{aligned}
N^{-1}\mathcal{L} = & |\partial_\mu\phi + igA_\mu\phi|^2 \\
& + \frac{1}{2\lambda}\chi^2 - \frac{\mu^2}{\lambda}\chi - \chi|\phi|^2 \\
& - \frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2\xi}Tr(\partial_\mu A^\mu)^2 \\
& - Tr\{\partial_\mu C^*(\partial^\mu C + ig[A^\mu, C])\} \\
& + i\sum_{i=1}^{N_f}(\bar{\psi}_i\gamma\cdot D\psi_i)
\end{aligned} \tag{2.1}$$

In (2.1)  $\phi$  and  $\psi_i$  transform in the fundamental representation of  $U(N)$ , the gauge field (ghost)  $A_\mu$  ( $C$ ) transform in the adjoint,  $\chi$  is a  $U(N)$  singlet, and  $D$  is the covariant derivative. The field  $\chi$  serves as a Lagrange parameter, which if eliminated, reproduces the usual  $\lambda\phi^4$  scalar self-interactions. [The coupling constants and fields have been rescaled so that  $N$  is an overall factor of

the Lagrangian, and hence  $1/N$  is a suitable expansion parameter.] Note that there is no Yukawa coupling between  $\phi$  and  $\psi$ , since both are in the fundamental representation. In the absence of the gauge couplings, (2.1) reduces to the usual vector model with  $U(N)$  symmetry [4, 5], together with  $N_f$  free fermions.

In this section we present the results of a calculation of the effective potential derived from (2.1) to leading order in  $1/N$ , to all orders in  $\lambda$ , and leading order in  $g^2$ . It is convenient to work in Landau gauge ( $\xi = 0$ ), so that the gauge parameter will not be renormalized. The resulting effective potential is renormalized using modified minimal subtraction, with the relevant relations between bare and renormalized quantities to the order indicated above being

$$\begin{aligned}
d &= 4 - 2\epsilon \\
M^\epsilon \phi_b &= Z_\phi^{1/2} \phi_r \\
M^\epsilon \psi_b &= Z_\psi^{1/2} \psi_r \\
\chi_b &= Z_\phi^{-1} \chi_r \\
M^{-\epsilon} g_b &= g_r \\
M^{2\epsilon} \left( \frac{\mu^2}{\lambda} \right)_b &= Z_\phi \left( \frac{\mu^2}{\lambda} \right)_r \\
M^{2\epsilon} \left( \frac{1}{\lambda} \right)_b &= Z_\phi^2 \left( \frac{1}{\lambda} \right)_r - \frac{1}{16\pi^2\epsilon} - \frac{1}{16\pi^2} \left( \frac{g^2}{16\pi^2} \right) \left( \frac{3}{\epsilon^2} + \frac{4}{\epsilon} \right) \\
Z_\phi &= 1 + \frac{g_r^2}{16\pi^2} \left( \frac{3}{\epsilon} \right). \tag{2.2}
\end{aligned}$$

The subscripts  $b$  and  $r$  refer to bare and renormalized quantities respectively, while  $M$  is an arbitrary renormalization mass-scale. [Note that in the  $1/N$  expansion it is natural to renormalize  $1/\lambda$  rather than  $\lambda$ .] The resulting renormalized effective potential is

$$N^{-1}V = \frac{\mu^2}{\lambda} \chi + \chi \phi^2 - \frac{1}{2\lambda} \chi^2$$

$$\begin{aligned}
&+ \frac{1}{16\pi^2} \chi^2 \left[ \frac{1}{2} \ln \left( \frac{\chi}{M^2} \right) - \frac{3}{4} \right] \\
&- \frac{1}{16\pi^2} \left( \frac{g^2}{16\pi^2} \right) \chi^2 \left[ \frac{3}{2} \ln^2 \left( \frac{\chi}{M^2} \right) - 7 \ln \left( \frac{\chi}{M^2} \right) + c \right]
\end{aligned} \tag{2.3}$$

where  $c$  is a numerical constant not relevant to our order. [The subscripts  $r$  will be omitted in all that follows. Note that the fermions do not contribute to (2.3) to leading order in  $N$  and  $g^2$ , due to the absence of a Yukawa coupling. For convenience we write  $\phi^2$  instead of  $|\phi|^2$ .] One is interested in the ultraviolet behavior of the effective potential to see if the difficulties found by Abbott *et al.*, [5] have been eliminated. However, when  $\chi/M^2 \gg 1$ , one encounters large logarithms which make (2.3) unreliable in that region. Therefore we consider the renormalization group improved effective potential, which will provide an effective potential which is independent of  $M$ , to the order we are working, and suppress the dependence on large logarithms.

To this order, we want an effective potential which satisfies

$$\begin{aligned}
0 &= M \frac{dV}{dM} \\
&= \left[ M \frac{\partial}{\partial M} + \beta_{1/\lambda} \frac{\partial}{\partial(1/\lambda)} + \beta_g \frac{\partial}{\partial g} \right. \\
&\quad \left. + \beta_{\mu^2/\lambda} \frac{\partial}{\partial(\mu^2/\lambda)} - \gamma_\phi \phi^2 \frac{\partial}{\partial \phi^2} + \gamma_\phi \chi \frac{\partial}{\partial \chi} \right] V,
\end{aligned} \tag{2.4}$$

and agrees with (2.3) when expanded in  $g^2$  and  $\ln(\chi/M^2)$ . Equation (2.4) does not depend on  $\psi$ , as there are no external fermion insertions. The  $\beta$ -functions and anomalous dimensions obtained from (2.2) are, to leading order in  $N$

$$\beta_{1/\lambda} = M \frac{d}{dM} \left( \frac{1}{\lambda} \right) = \frac{1}{16\pi^2} \left[ \frac{32\pi^2\epsilon}{\lambda} + \frac{12g^2}{\lambda} - 2 - \frac{g^2}{\pi^2} \right] \tag{2.5a}$$

$$\beta_g = M \frac{d}{dM} g = -\epsilon g - g^3/16\pi^2 \left( \frac{22}{3} - \frac{4}{3} \frac{N_f}{N} \right) + \mathcal{O}(g^5) \tag{2.5b}$$

$$\beta_{(\mu^2/\lambda)} = M \frac{d}{dM} \left( \frac{\mu^2}{\lambda} \right) = (2\epsilon - \gamma_\phi) \left( \frac{\mu^2}{\lambda} \right) \tag{2.5c}$$

$$\gamma_\phi = M \frac{d}{dM} \ln Z_\phi = \frac{-6g^2}{16\pi^2} + \mathcal{O}(g^4). \tag{2.5d}$$

We therefore consider  $N_f$  of order  $N$ . It is useful to define

$$\gamma' = \frac{\gamma_\phi}{(1 - \gamma_\phi/2)}$$

so that  $\chi^2 (\frac{\chi}{M^2})^{\gamma'}$  is a renormalization group invariant. Note that the conventional  $\beta_\lambda$  function is related to (2.5a) by

$$\beta_\lambda = -\lambda^2 \beta_{1/\lambda}$$

so that

$$16\pi^2 \beta_\lambda = -32\pi^2 \epsilon \lambda + 2\lambda \left( \lambda - 6g^2 + \frac{g^2 \lambda}{2\pi^2} \right). \quad (2.6)$$

Let  $M_0$  be the mass-scale at which the coupling constants and “composite” field  $\chi$  are defined, and

$$\begin{aligned} \lambda_0 &= \lambda(M_0) \\ \chi_0 &= \chi(M_0; \phi^2) \end{aligned} \quad (2.7)$$

### 3. Renormalization Group Improved Effective Potential

#### A. Effective Potential

Let us consider the renormalization group improvement of the effective potential (2.3)

From (2.5b)

$$g^2(M) = g_0^2 \left[ 1 + \frac{4}{3} \left( 11 - 2 \frac{N_f}{N} \right) \frac{g_0^2}{16\pi^2} \ln \left( \frac{M}{M_0} \right) \right]^{-1} \quad (3.1)$$

where

$$g_0 = g(M_0). \quad (3.2)$$

From (2.5a) and (2.5b) we have

$$M \frac{d}{dM} \left[ \frac{16\pi^2}{\lambda} - \left( \frac{2}{12 - A} \right) \frac{16\pi^2}{g^2} \right] = \frac{12g^2}{16\pi^2} \left[ \frac{16\pi^2}{\lambda} - \left( \frac{2}{12 - A} \right) \frac{16\pi^2}{g^2} \right] + \mathcal{O}(g^2) \quad (3.3)$$

where

$$A = \frac{4}{3} \left( 11 - 2 \frac{N_f}{N} \right) \quad (3.4)$$

We see that there is a phase-boundary when

$$\lambda = \frac{4}{3} \left( \frac{N_f}{N} - 1 \right) g^2 \quad (3.5)$$

A graph of  $\lambda$  versus  $g^2$  is shown in Figure 1, with the renormalization group flows indicated on the graph. Note the two-phase structure of the theory, with the line  $\lambda = 4/3 (\frac{N_f}{N} - 1)g^2 = C g^2$  in the  $(\lambda, g^2)$  plane separating the two-phases. For  $\lambda < C g^2$ , the theory is asymptotically free, while if  $\lambda > C g^2$ , the theory is not, since  $\lambda \rightarrow \infty$  in the ultraviolet in this phase even though  $g^2 \rightarrow 0$ . Thus the qualitative ultraviolet behavior of the theory in the phase  $\lambda > C g^2$  is similar to that of the ungauged theory. If the initial conditions for the renormalization group are chosen such that  $\lambda_0 = C g_0^2$ , then  $\lambda/g^2 = C$  throughout the renormalization group flow, to the order we are working. Note that if  $g_0^2/16\pi^2 \ll 1$ , then also  $\lambda_0/16\pi^2 \ll 1$  in the asymptotically free phase; which *à posteriori* is in the domain of perturbation theory.

The solution to (2.4) and (2.5), with (2.3) as boundary conditions, gives the renormalization group improved effective potential<sup>3</sup>

$$\begin{aligned} N^{-1}V &= \frac{\mu^2}{\lambda} \chi + \chi \phi^2 \\ &- \frac{1}{2} \chi^2 \left\{ \frac{1}{\lambda} - \left( \frac{2}{12-A} \right) \frac{1}{g^2} \left[ 1 - \left( 1 + \frac{A}{2} \frac{g^2}{16\pi^2} \ln \left( \frac{\chi}{M^2} \right) \right)^{\frac{A-12}{A}} \right] \right\} \end{aligned} \quad (3.6)$$

where  $A$  is given by (3.4). In solving the renormalization group equation (2.4) one matches only the leading logarithms of (2.3), since one has no control of the sub-leading logarithms, to the order

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<sup>3</sup>The condition for broken symmetry,  $\mu^2/\lambda < 0$  with  $\frac{\partial V}{\partial \phi^2} = 0$  requires  $\chi = 0$  in both the tree level potential, and in (3.5). Also, the massless fermions do not contribute to the vacuum energy to leading order in  $N$  and  $g^2$ . Therefore, the vacuum energy is zero and does not need separate renormalization group improvement. See [18]. We thank B. Kastening for raising this point.

we are working. Thus  $M \frac{dV}{dM}$  is not identically zero for (3.6), but is zero to the order of accuracy required of our approximations. The gap equation  $\partial V/\partial\chi = 0$  means that

$$\begin{aligned} \phi^2 = \frac{-\mu^2}{\lambda} + \chi & \left\{ \frac{1}{\lambda} - \left( \frac{2}{12-A} \right) \frac{1}{g^2} \left[ 1 - \left( 1 + \frac{A}{2} \frac{g^2}{16\pi^2} \ln \left( \frac{\chi}{M^2} \right) \right)^{\frac{A-12}{A}} \right] \right. \\ & \left. - \frac{1}{32\pi^2} \left[ 1 + \frac{A}{2} \frac{g^2}{16\pi^2} \ln \left( \frac{\chi}{M^2} \right) \right]^{\frac{-12}{A}} \right\} \end{aligned} \quad (3.7)$$

Thus, inserting (3.7) into (3.6)

$$N^{-1}V = \frac{1}{2} \chi^2 \left\{ \frac{1}{\lambda} - \left( \frac{2}{12-A} \right) \frac{1}{g^2} \left[ 1 - \left( 1 + \frac{A}{2} \frac{g^2}{16\pi^2} \ln \left( \frac{\chi}{M^2} \right) \right)^{\frac{A-12}{A}} \right] \right\}, \quad (3.8)$$

where we dropped subleading terms in  $g^2 \ln(\chi/M^2)$ , which can be neglected to the order we are working. The coefficient of the subleading term in (3.7) is not determined by our computation, because it corresponds to a higher order term in (3.6). However, since what we really have is (3.6) and  $\partial\chi/\partial V = 0$ , (3.7) must be used as shown in numerical computations. One must resort to a numerical evaluation for  $V(\phi^2)$  since (3.6) with (3.7) cannot be evaluated analytically.

Note from Figure 1 that the model has two phases, with the phase boundary given by<sup>4</sup>

$$0 < \lambda = \frac{4}{3} \left( \frac{N_f}{N} - 1 \right) g^2 \quad (3.9)$$

which depends on  $1 < (\frac{N_f}{N}) < \frac{11}{2}$ , where the upper-bound is required so as to maintain asymptotic freedom, as can be seen from (3.1). Since  $g^2(M)$  has a Landau singularity in the infrared region, a Landau pole appears in the infrared region of the effective potential. The infrared pole should be regarded as a signal of the confinement of colored degrees of freedom, and not a fundamental flaw in the model. In the asymptotically free phase, where  $\lambda < \frac{4}{3}(\frac{N_f}{N} - 1)g^2$ , the effective potential has a lowest energy bound in the ultraviolet, while if  $\lambda > \frac{4}{3}(\frac{N_f}{N} - 1)g^2$  there is no lowest energy

<sup>4</sup>Note that there is no asymptotically free phase for  $N_f = 0$ . A factor of two error in (3.1) led to the opposite conclusion in an earlier version of this paper.

bound in the ultraviolet. [With the presence of the Landau pole, one cannot discuss a lowest energy bound in the infrared in a meaningful way.] We present in Figs. 2 and 3, the effective potential  $V(\phi^2)$  for the two phases, for  $\mu^2 < 0$ . The infrared singularity occurs in a range of  $\phi^2$  many orders of magnitude smaller than the scale of the figures, so it disappears in the interval between two of the numerically computed points. The figures emphasize the possibility of spontaneous symmetry breaking, and stability or lack of stability of the phases. [In Figs. 2 and 3,  $g^2(M_0)/16\pi^2$  is chosen sufficiently small so that the Landau pole is in the extreme infrared region.]

The phase boundary is given by (3.8), with corrections of  $\mathcal{O}(g^4)$  expected. For  $g^2(M_0)/16\pi^2$  sufficiently small, these corrections are not expected to shift the phase-boundary in any significant way. We observe from (3.3) and (3.4) that the **ratio**  $\lambda/g^2$  does not run at the phase boundary, although both  $g^2(M^2)$  and  $\lambda(g^2)$  flow to zero in the ultraviolet.

## B. Vector Meson Spectrum and Confinement

As we have just discussed, Figs. 2 and 3 give a description of the effective potential for our model for  $\mu^2 < 0$ , in the two phases of the theory. The results for  $\mu^2 > 0$  are qualitatively similar, except the  $\phi^2 = 0$  axis in Figs. 2 and 3 is shifted to the right, so that no spontaneous symmetry breaking takes place. The Landau singularity, which is not shown in the figures, signals that the theory likely will confine.

It is claimed that in non-abelian gauge theories with matter in the fundamental representation, one evolves from a “Higgs” description of the theory to a confining description, as the parameters of the model are changed, **without** encountering a phase transition [19]. A logical choice for  $M_0$  is  $M_0 \simeq \mu$ , as  $\mu$  is the only mass-scale in the problem. Then the magnitude of  $g^2(\mu)/16\pi^2$  determines whether a “Higgs” or confinement description is more appropriate. In our case, this corresponds to the evolution from  $g^2(\mu)/16\pi^2 \ll 1$  to larger values of  $g^2(\mu)/16\pi^2$ . Thus

Fig. 2 is appropriate to the Higgs description of the theory, since the Landau singularity obtained from running  $g^2$  appears in the far infrared region.

When  $g^2(\mu)/16\pi^2$  small and  $\mu^2 < 0$ , the asymptotically free phase leads to a vector meson spectrum that is well described by perturbative estimates of masses, *i.e.*,  $N(N-2)$  massless vectors and  $2N-1$  massive vectors with  $M_v^2 \sim 2g^2v^2$ . If on the other hand  $g^2(\mu^2)/16\pi^2$  is large, then the vector mesons are strongly interacting at the characteristic mass-scale  $\mu$ , so that the confinement description would be more appropriate. In analogy with the quark model, one then might wish to assign “short-distance” masses to the vectors, and run them by means of a renormalization group. However, this is well outside the scope of our calculation. In the asymptotically free phase, but with  $\mu^2 > 0$ , the gauge bosons should be regarded as massless, and confined.

The methods of this paper demonstrate that the two-phase structure is essential for understanding the physics.

### C. Phase Transition in $N$ ?

The question arises as to whether these results are stable as  $N$  is decreased.<sup>5</sup> We do not have the tools to answer this question in general, as our calculation is to all orders in  $\lambda$ , but only leading orders in  $1/N$  and  $g^2$ . The complete phase surface is inaccessible as it depends to all orders on  $\lambda$ ,  $g^2$ , and  $1/N$ . We can provide a very limited answer to this question by including the known  $1/N$  corrections to the  $\beta$ -functions to leading order in  $g$ . Then instead of (2.5a) and (2.5b), we have [14, 15]

$$16\pi^2\beta_{g^2} = -g^4 \left( \frac{44}{3} - \frac{8}{3} \frac{N_f}{N} - \frac{2}{3N} \right) + \mathcal{O}(g^6) \quad (3.10a)$$

$$16\pi^2\beta_{1/\lambda} = 12 \left( 1 - \frac{1}{N^2} \right) \frac{g^2}{\lambda} - 2 \left( 1 + \frac{4}{N} \right) . \quad (3.10b)$$

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<sup>5</sup>We thank the referee for raising this question.

It is straightforward to show that the phase-boundary determined by (3.10) is

$$0 < \lambda = \frac{\left[ \frac{4}{3} \left( \frac{N_f}{N} - 1 \right) + \frac{1}{3N} - \frac{6}{N^2} \right] g^2}{\left( 1 + \frac{4}{N} \right)} \quad (3.11)$$

while asymptotic freedom for  $g^2$  requires

$$\left( \frac{11}{2} - \frac{1}{4N} \right) > \frac{N_f}{N} . \quad (3.12)$$

Thus, if  $N_f$  is fixed (and very large), then as  $N$  is decreased (3.12) will eventually be violated, and asymptotic freedom will be lost. On the other hand, if one keeps  $(\frac{N_f}{N})$  fixed, so that (3.12) is satisfied, then for  $(\frac{N_f}{N}) > 2$ , (3.11) can be satisfied for  $\lambda > 0$  for any  $N$ , so that the two-phase structure of the model exhibited in the large  $N$  limit can be preserved.

#### 4. Gauge Invariance

It is known for some time that the effective potential is not gauge invariant [16, 17]. How does this impact our results? We address this issue in this section.

Consider the effective action  $\Gamma(\phi, \chi, A_\mu)$ , and

$$V(\bar{\phi}, \bar{\chi}, \bar{A}_\mu) = - \int d^4x \Gamma(\bar{\phi}, \bar{\chi}, \bar{A}_\mu) / \int d^4x \quad (4.1)$$

when evaluated at the stationary points  $\bar{\phi}, \bar{\chi}, \bar{A}_\mu$  defined by

$$\frac{\delta \Gamma}{\delta \phi} = \frac{\delta \Gamma}{\delta \chi} = \frac{\delta \Gamma}{\delta A_\mu} = 0 . \quad (4.2)$$

In this paper we have considered  $V(\bar{\phi}, \bar{\chi}, 0)$ , which is obtained from

$$\left( \frac{\delta \Gamma}{\delta \phi} \right)_{A_\mu=0} = \left( \frac{\delta \Gamma}{\delta \chi} \right)_{A_\mu=0} = 0 \quad (4.3)$$

which need not be the stationary points of (4.2). However it has been shown that by Fukuda and Kugo [17] that there are a wide class of “good gauges” where  $V(\bar{\phi}, \bar{\chi}, 0)$  correctly describes the stationary points of (4.2) by means of (4.3) with  $\bar{A}_\mu = 0$ . The “good gauges” include covariant

gauges, Landau gauge,  $R_\xi$  gauge, axial gauge .... [By contrast in a bad gauge,  $\bar{A}_\mu(x) \neq 0$  at the stationary points, and  $\bar{\phi}(x)$  has  $x$  dependence to compensate that of  $\bar{A}_\mu(x)$  and restore Lorentz invariance.] The Landau gauge employed in this paper is a good gauge, for which the stationary points are gauge invariant.

In more detail, it can be shown [17] that the total variation with respect to the gauge parameter satisfies a renormalization group type equation, where schematically

$$\begin{aligned} \frac{D}{D\alpha} V(\phi, \alpha) &= \left[ \frac{\partial}{\partial\alpha} - \gamma_\lambda^{(\alpha)} \left( \lambda \frac{\partial}{\partial\lambda} \right) - \gamma_\mu^{(\alpha)} \left( \mu \frac{\partial}{\partial\mu} \right) \right] V(\phi, \alpha) \\ &= \left( \frac{\partial V}{\partial\phi_i} \right) F_i(\phi, \alpha, A_\mu) \quad (i = 1 \text{ to } N) \end{aligned} \quad (4.4)$$

where  $F(\phi, \alpha, A_\mu)$  is a functional of the fields and

$$\gamma^{(\alpha)} = Z \frac{\partial}{\partial\alpha_0} \ln Z \quad (4.5)$$

for the two anomalous dimensions. Thus

$$\frac{D}{D\alpha} V(\phi, \alpha) = 0 \quad \text{at } \phi = \bar{\phi}. \quad (4.6)$$

This means that the explicit gauge dependence cancels the implicit gauge dependence of the parameters ( $\mu^2/\lambda$ ) and  $\lambda$  at the critical point  $\bar{\phi}$ . Therefore, the value of the effective potential is gauge invariant at the critical points, so that one can select the critical point with the lowest value of the effective potential  $V$  in a gauge invariant way [16]. [Spontaneous symmetry breaking is a gauge invariant concept.]

Further,

$$\begin{aligned} \frac{D}{D\alpha} \left[ \frac{\partial V(\phi, \alpha)}{\partial\phi_j} \right] &= \frac{\partial^2 V}{\partial\phi_i \partial\phi_j} F_i(\phi, \alpha, A_\mu) \\ &+ \frac{\partial V}{\partial\phi_i} \frac{\partial F_i}{\partial\phi_j} (\phi, \alpha, A_\mu) \end{aligned} \quad (4.7)$$

$$\begin{aligned}
&= \left[ 2\delta_{ij} \left( \frac{\partial V}{\partial \phi^2} \right) + 4\phi_i \phi_j \left( \frac{\partial^2 V}{(\partial \phi^2)^2} \right) \right] F_i(\phi, \alpha, A_\mu) \\
&+ \left[ 2\phi_i \left( \frac{\partial V}{\partial \phi^2} \right) \right] \frac{\partial F_i(\phi, \alpha, A_\mu)}{\partial \phi_j} .
\end{aligned} \tag{4.8}$$

At the phase-boundary, both

$$\left( \frac{\partial V}{\partial \phi^2} \right)_{\bar{\phi}} = 0 \quad \text{and} \quad \left( \frac{\partial^2 V}{(\partial \phi^2)^2} \right)_{\bar{\phi}} = 0 \tag{4.9}$$

Hence at the phase boundary, one also has at the stationary point of  $V$ ,

$$\frac{D}{D\alpha} \left[ \frac{\partial V}{\partial \phi_j} \right]_{\bar{\phi}} = 0 . \tag{4.10}$$

This means that at the phase-boundary, not only is  $V(\bar{\phi})$  gauge invariant, but  $\left( \frac{\partial V}{\partial \phi} \right)_{\bar{\phi}}$  is as well.

Thus, the vanishing of  $dV/d\phi^2$  evaluated at  $\bar{\phi}$  at the phase-boundary is a gauge invariant criterion, as expected for a zero-mass bound-state [16].

In general, the effective potential is not gauge invariant [16, 17] so that the effective potential need not have the specific behavior of Figs. 2 and 3 in other gauges. However, the separation of the theory into asymptotically free and non-asymptotically free phases is a gauge invariant concept. Thus one expects the resolution of the difficulties of the ungauged vector model provided by the asymptotically free phase to be a physical feature of the model.

## Conclusions

We have presented a calculation of the renormalization group improved effective potential for the gauged vector model coupled to  $N_f$  massless fermions in the defining representation, computed to leading order in  $1/N$ , all orders in  $\lambda$ , and leading order in  $g^2$ . It was shown that the theory has two phases. In the asymptotically free phase, the effective potential behaves qualitatively like that of the tree-approximation, but with a Landau pole in the infrared region. If  $\lambda$  is too large, asymptotic freedom is destroyed, and the effective potential exhibits all the difficulties found previously for the ungauged theory ( $g^2 = 0$ ) [5].

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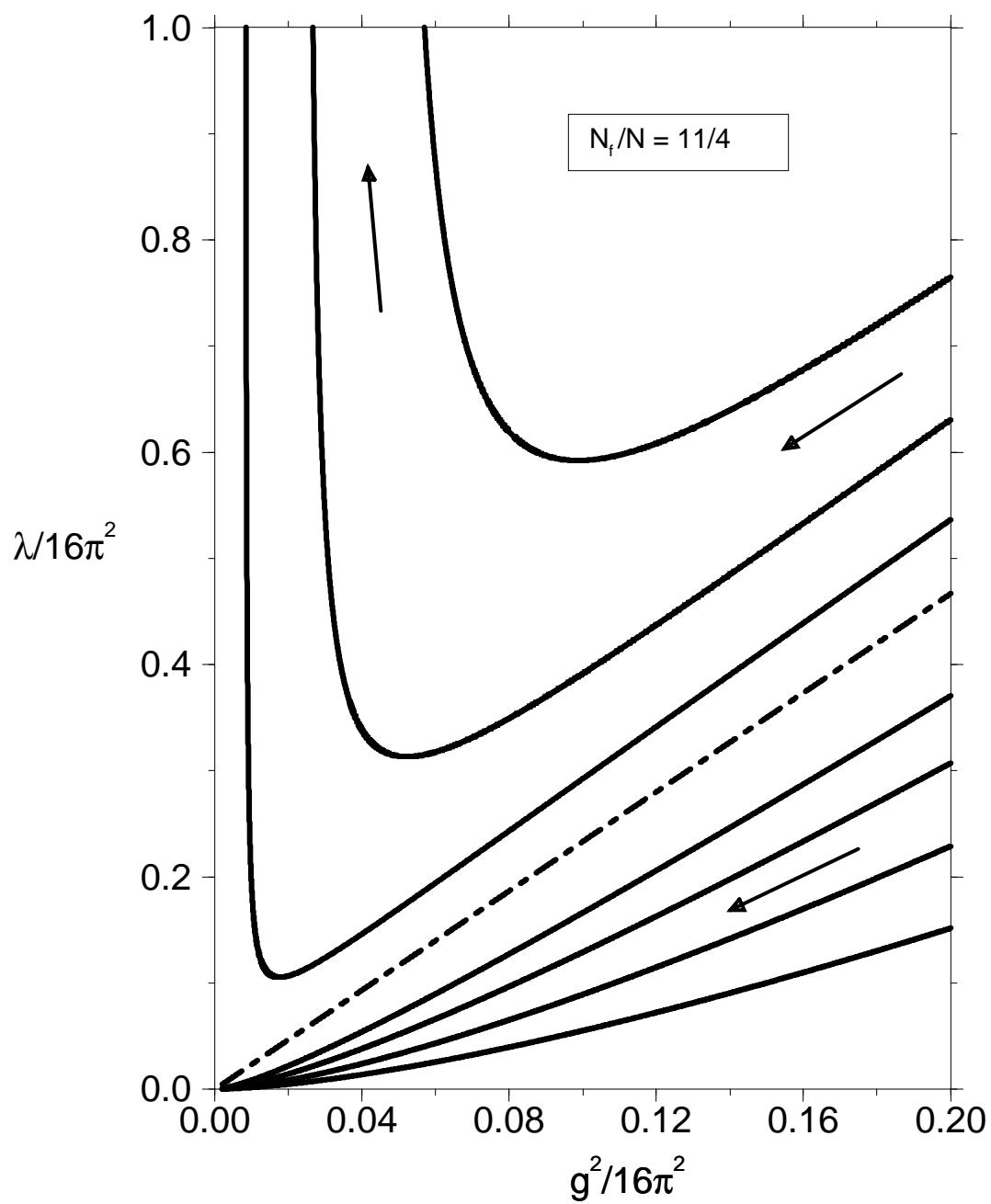
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## Figure Captions

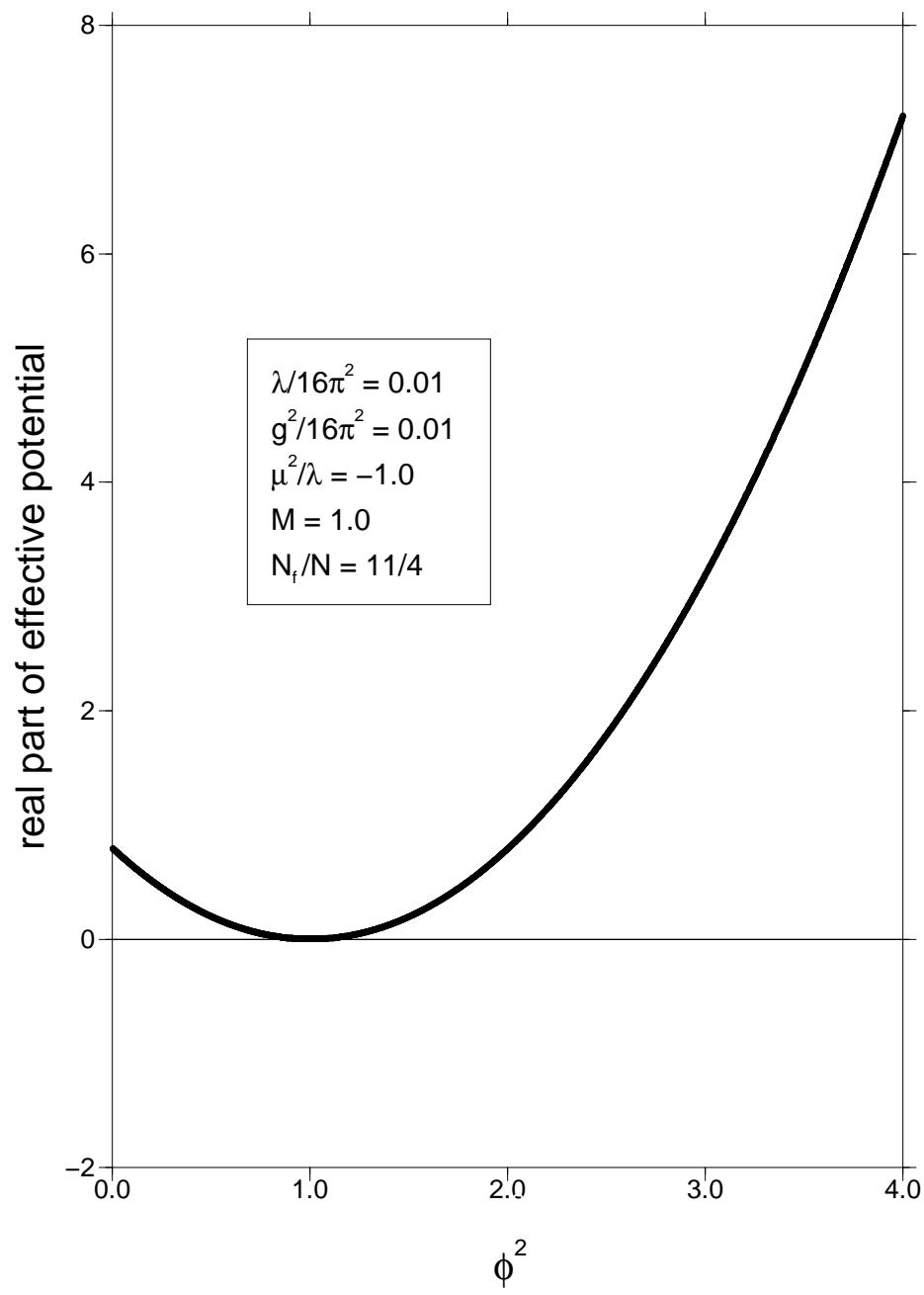
**Fig. 1:** The  $\lambda$  versus  $g^2$  plane, and renormalization group flows. Note the two phase structure. The arrows point toward the ultraviolet. The phase boundary is  $\lambda = \frac{4}{3}(\frac{N_f}{N} - 1)g^2$ . The graph is for  $\frac{N_f}{N} = \frac{11}{4}$ .

**Fig. 2:** The real part of the effective potential vs.  $\phi^2$  for  $g^2/16\pi^2 = 0.01$ ,  $\lambda/16\pi^2 = 0.01$ ,  $\mu^2/\lambda = -1$ , and  $M = 1$ , for  $\frac{N_f}{N} = \frac{11}{4}$ . In both Figs. 2 and 3, there are singularities (Landau poles) in  $V$ , near  $\phi^2 = 1$ ,  $V = 0$ . These features are not visible in the figures because they occur in a range of  $\phi^2$  which is many orders of magnitude smaller than the scale displayed.

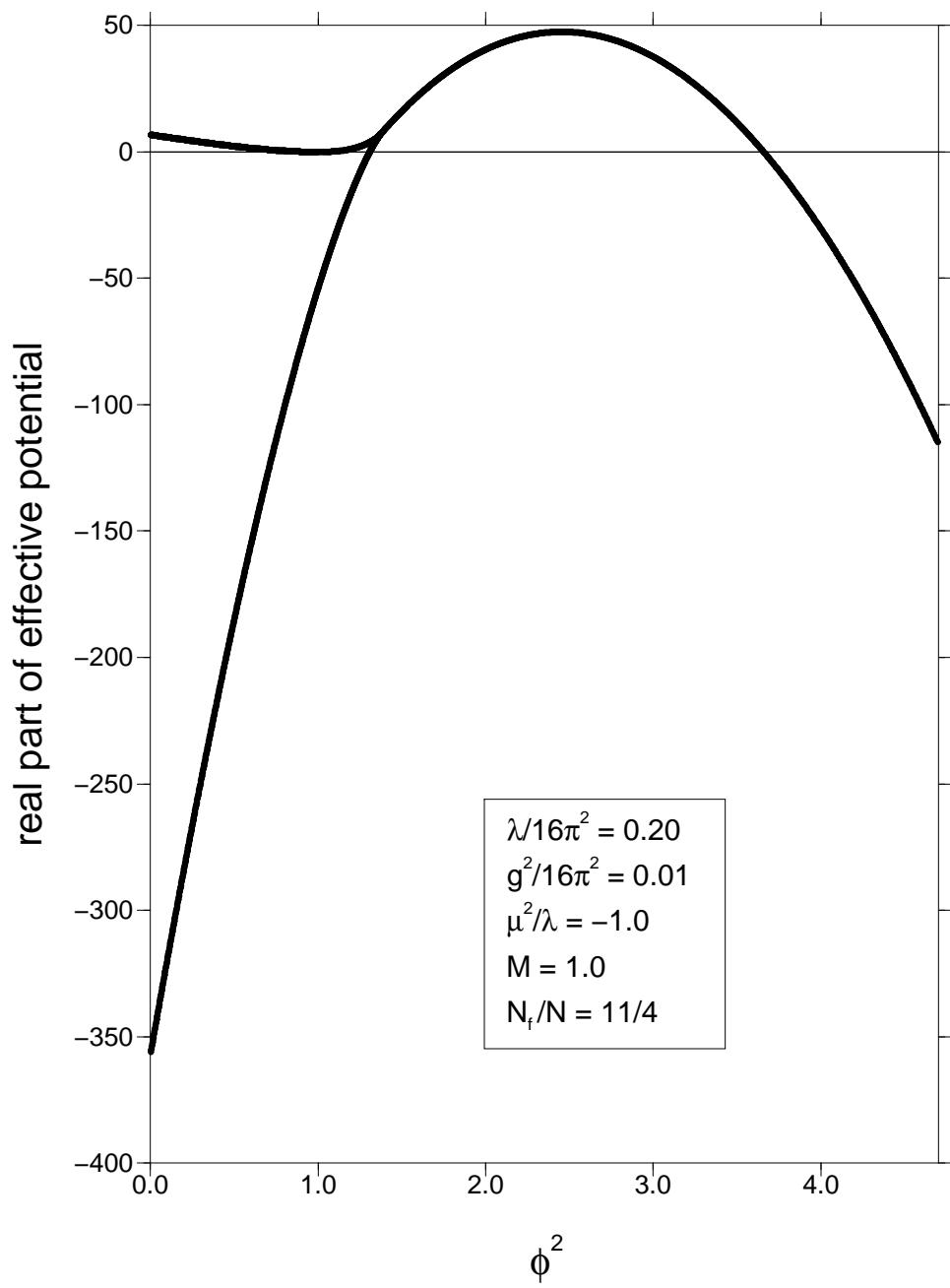
**Fig. 3:** Same as Fig. 2, except  $\lambda/16\pi^2 = 0.20$ .



• Figure 1



• Figure 2



• Figure 3